Transmit diversity versus SDMA: analytic and numerical comparisons

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Abstract—**In this paper, the capacity of a wireless link with multiple transmit antennas is compared to the average per user capacity of a multiuser wireless network with Space Division Multiple Access (SDMA). Under the assumption of a rich scattering environment and a narrow-band link, the propagation medium is modeled as a Rayleigh flat fading with a good receive diversity. By contrast to the previous works, the assumption of full decorrelation of the transmit antennas is relaxed. This enables a study of the transmit diversity scenarios where the transmit antennas occupy a limited space. The new analytic results give rise to a comparison of the capacity achievable with multiple transmit antennas within a limited transmit volume on one hand, and the capacity of an SDMA network with spatially spread users, on the other hand. The capacity simulations corresponding to indoor wireless LAN are presented along with the theoretical results.**

I. Introduction

The uplink channel capacity in SDMA has been studied by the authors of [1] in the case of a deterministic propagation channel specified by the powers and directions of arrival of different users as seen at the access point (base station). Despite some meaningful results regarding the capacities achievable with different decoding schemes, the statistical properties (*e.g.*, the outage probability) of the SDMA channels driven by the commonly used random models were not discussed in the aforementioned paper. More recently, Foschini *et al.* studied the capacity of a narrow-band wireless link between multiple transmit and receive antennas and nearly optimal transmission schemes when the propagation channel is assumed Rayleigh and flat with *i.i.d.* coefficients [2]. Accurate capacity bounds with and without channel knowledge and the asymptotic statistical properties of the optimally shaped data blocks are available in [3]. These new results promise extraordinary capacities when the number of the transmit antennas is big enough (assuming that there are at least as many receive antennas as the transmit ones). More precisely, the capacity was shown to scale linearly with the number of the transmit antennas when the coefficients of the channel matrix are *i.i.d.* The analysis presented in [2], [3] applies to the SDMA systems as well as to the transmit diversity case where multiple transmit antennas are used. Based on the assumption of rich scattering environment, this analysis implies equal capacities in SDMA and transmit diversity scenarios. However, SDMA systems have a higher potential since space diversity of different users is generically better than the diversity of multiple transmit antennas when the transmit volume (*i.e.*, the space occupied by these antennas) is relatively small. Such a mismatch comes from inadequate modeling of random vector-valued channels between each transmit antenna and the set of receive antennas as statistically independent values.

The independence condition is relaxed in this contribution so that the channels corresponding to different transmit antennas may exhibit an arbitrary correlation. It is still assumed that the receive antennas are decorrelated. This assumption appears to be quite accurate for the uplink channel scenarios with no line-of-sight (LOS). The signal captured at the access point of a WLAN result from a rich local scattering while the wireless terminal is relatively far away. In such a case, decorrelation of the receive antennas is due to a spatially uniform spread of scattering points w.r.t. the access point. Meanwhile, major part of the scatterers from the receiver vicinity stay within a limited sector as seen by the transmit antennas. For a relatively big distance between the transmitter and the receiver, a small sector gives rise to a non-negligible correlation of the transmit antennas.

For this model, tight capacity bounds are calculated. A rather general form of the transmit diversity matrix is derived under the assumption of the spatially uniform spread of scattering points in the vicinity of the receiver. This matrix is used along with the capacity bound to characterize analytically the potential of transmit diversity schemes as compared to the SDMA systems. Theoretical capacity bounds are presented together with the empirical capacities computed for uplink channels of typical

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indoor WLAN scenarios.

II. Data model

Consider a flat fading channel between m transmit and M receive antennas such that

$$x_t = H\, s_t + n_t, \qquad t \in \mathbb{Z}, \qquad (1)$$

where s_t is the $m \times 1$ vector of the transmit antenna outputs, x_t is the $M \times 1$ vector of the received signals, H is the $M \times m$ channel matrix and n_t is the $M \times 1$ vector of the observation noise. Assume that

As1 Channel noise is a complex circular AWGN of power σ^2: $\mathbb{E}\{n_t\, n_t^H\} = \sigma^2\, I_M$.

As2 Each s_t is *i.i.d.* These signals are correlated and the total power is fixed:

$$\mathbb{E}\{s_t\, s_t^H\} = \sigma_s^2\, R_s, \qquad \mathrm{tr}\,(\, R_s\,) = 1.$$

Denote by $\varrho^2 = (\sigma_s^2/\sigma^2)$ the overall signal-to-noise ratio (SNR). According to [4], the channel capacity (in bits per second per hertz) for this scenario is given by

$$C = \log_2 \det \left(\, I_M + \varrho^2\, H\, R_s\, H^H\, \right). \qquad (2)$$

The Rayleigh channel model is assumed so that the entries of H are jointly complex circular Gaussian:

$$\mathrm{vec}\,\{H\} \sim \mathcal{N}_c(\,0, \Sigma\,), \qquad (3)$$

where vec $\{\cdot\}$ denotes stacking of matrix' columns to a column vector and Σ is an $Mm \times Mm$ normalized correlation matrix (*i.e.*, $\Sigma_{kk} = 1$, $1 \le k \le Mm$). In general, it is natural to assume that on one hand, the contributions of all transmit antennas to the receive antenna array are statistically equivalent and that, on the other hand, the contributions of the transmit array to all receive antennas are statistically equivalent. In other words,

$$\Sigma = R_T \otimes R_R, \qquad (4)$$

where $(\otimes)$ denotes the Kronecker product and R_T (R_R) stands for the transmit (receive) correlation matrix. As already mentioned, full decorrelation of the receive antennas is assumed, *i.e.*, $R_R = I_M$ is the $M \times M$ identity matrix. The transmit diversity, however, depends on the ratio of the transmit array size to the distance between the transmitter and the receiver positions. When this ratio goes to zero, R_T tends to a rank-one matrix. Indeed, the transmit array is seen by the set of the receive array (or by any remote scatterer) as a single point when the distance between the arrays becomes infinite.

III. Lower bound for channel capacity

Define $H = \underline{H}\, R_T^{\frac{1}{2}}$, then the entries of $\underline{H}$ are *i.i.d.* $\mathcal{N}_c(0,1)$. Define also $\{U, \Lambda^2\}$ the eigendecomposition of $(R_T^{\frac{1}{2}} R_s R_T^{\frac{1}{2}})$ so that $R_T^{\frac{1}{2}} R_s R_T^{\frac{1}{2}} = U \Lambda^2 U^H$ with a diagonal matrix $\Lambda = \mathrm{diag}\{\Lambda_k\}_{k=1}^m$ and a unitary U. Now, (2) may be rewritten as follows:

$$\begin{aligned} C &= \log_2 \det \left(\, I_M + \varrho^2\, \underline{H}\, \Lambda^2\, \underline{H}^H\, \right) \\ &= \log_2 \det \left(\, I_m + \varrho^2\, \Lambda\, \underline{H}^H \underline{H}\, \Lambda\, \right). \end{aligned}$$

Theorem 1: There exist m statistically independent variables $\chi^2_{M-1}, \chi^2_{M-2}, \cdots \chi^2_{M-m+1}$ and m non-negative variables $\vartriangle C_1, \vartriangle C_3, \ldots, \vartriangle C_m$ such that $\vartriangle C_1 = 0$, $\vartriangle C_k = 0$ when $\Lambda_k = 0$ and

$$\begin{aligned} C &= \sum_{k=1}^{m} \log_2 \left(\, 1 + \varrho^2 \Lambda_k^2\, \chi^2_{M-k+1}\, \right) + \vartriangle C_k, \\ \mathbb{P}\left\{\chi_k^2 < z\right\} &= \Gamma_x(\, z;\, k\,), \\ \mathbb{P}\left\{\vartriangle C_k > z\right\} &< \mathcal{B}_x(\, 2^{-z};\, M+1-k,\, k-1\,), \end{aligned}$$

where $\Gamma_x(\, z;\, k\,)$ is the *incomplete Gamma function*, $k \in \mathbb{Z}_+$, and $\mathcal{B}_x(\, z;\, a,\, b\,)$ is the *incomplete Beta function*, $a, b \in \mathbb{Z}_+$. If $\Lambda_1 \ge \ldots \ge \Lambda_m$, then

$$\mathbb{P}\left\{\vartriangle C_k > z\right\} < 1 - \Gamma_x(\,(\,2^z - 1\,)\,/\,(\,\varrho^2\, \Lambda_k^2\,);\, k-1\,).$$

According to theorem 1, we may define a lower bound on the channel capacity as follows:

$$C_* \triangleq \sum_{k=1}^{m} \log_2 \left(\, 1 + \varrho^2 \Lambda_k^2\, \chi^2_{M-k+1}\, \right), \qquad (5)$$

where $\Lambda_1 \ge \ldots \ge \Lambda_m$ and the random quantities χ^2_{M-k+1} are Gamma distributed. Theorem 1 yields tightness of the bound C_* at high and moderate SNR and big M. This claim is validated numerically; it suggests the use of the bound C_*.

The stochastic bound (5) also admits the following deterministic approximation.

$$C_\infty \triangleq \sum_{k=1}^{m} \log_2 \left(\, 1 + \varrho^2 \Lambda_k^2\, (M + 1 - k)\right), \qquad (6)$$

$\Lambda_1 \ge \ldots \ge \Lambda_m$. By using Jensen's inequality, one can find $\mathbb{E}\{C_*\} \le C_\infty$. A direct numerical evaluation shows that the relative error of this approximation is always less than 20%; it is actually much less than that in practical situations. The expressions (5) and (6) will be used to approximate capacities of SDMA and transmit diversity schemes.

IV. Limited transmit diversity

As it follows from the previous section, the capacity of a multiple antenna link depends on the eigenspectrum of R_T. Let us study the structure of R_T when the transmit array size is limited.

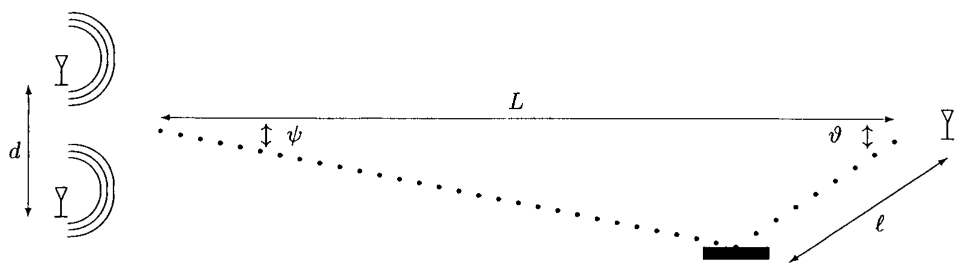

Fig.1. Propagation channel model

Let us calculate the covariance between the signals coming from two d-spaced antennas fed by the same input and collected by a receive antenna. These signals are scattered by an object which is ℓ far from the receive antenna, as shown in Fig.1. The distance between the transmitter and the receiver is L. Assume $d \ll L$ and $\ell \ll L$. The second condition reflects the fact that the energy captured by the receiver decreases along with the growth of ℓ. The aforementioned covariance may be approximated by

$$\gamma_s^2(L,\ell,\vartheta)\, e^{\imath 2\pi \frac{d}{\lambda}\sin\psi} \approx \gamma_s^2(L,\ell,\vartheta)\, e^{\imath 2\pi \frac{d}{\lambda}\frac{\ell \sin\vartheta}{L}}, \quad (7)$$

where $\gamma_s^2(L,\ell,\vartheta)$ stands for the mean power of the received signal and λ is the wavelength. Based on $\ell \ll L$, we may conclude that the intensity of the transmitted electromagnetic field is approximately the same for all local scattering points; it depends on L. The portion of the scattered energy that reaches the receiver depends on ℓ and ϑ. We will assume quite a general case of scattering models such that the density and the shape of scattering points is isotropic, *i.e.*, independent on the direction ϑ. Then $\gamma_s^2(L,\ell,\vartheta)$ may be written as follows:

$$\gamma_s^2(L,\ell,\vartheta) = \sigma_s^2\, \rho(\ell/L), \qquad \int_0^\infty \rho(u)\, du = 1, \quad (8)$$

where $\rho(u)$ is the normalized density of the scattered power captured by the receiver corresponding to the distance ℓ between the receiver and the scattering point measured in the units of L. The power σ_s^2 depends on the distance L and the path loss conditions. Define $\tau = d/\lambda$ the distance between the transmit antennas measured in wavelengths. Then the covariance in (7) versus τ averaged over (u,ϑ) is given by

$$\begin{aligned} c(\tau) &= \sigma_s^2 \int_0^\infty \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\imath 2\pi\, \tau\, u\, \sin\vartheta}\, d\vartheta \right) \rho(u)\, du \\ &= \sigma_s^2 \int_0^\infty J_0\left(2\pi\, \tau\, u\right)\, \rho(u)\, du, \qquad (9) \end{aligned}$$

where $J_0(\cdot)$ stands for the Bessel function. The last equality in (9) is due to (8.411-1) from [5]. The relative units τ and u are used rather than the natural distances in order to obtain scale-invariant results. As shown below, these quantities allow us to characterize the eigenstructure of $\boldsymbol{R}_T$. To get a better insight into the properties of the function $c(\tau)$, we will consider its Fourier transform

$$\underline{c}(\nu) = \int_{-\infty}^{\infty} c(\tau)\, e^{-\imath 2\pi\, \nu\, \tau}\, d\tau, \qquad \nu \in \mathbb{R}.$$

Substituting (9) into the last expression and using (6.671-8) from [5], we obtain

$$\underline{c}(\nu) = \frac{\sigma_s^2}{\pi} \int_{|\nu|}^{\infty} \frac{\rho(u)}{\sqrt{u^2 - \nu^2}}\, du, \qquad \nu \in \mathbb{R}. \quad (10)$$

According to the channel model, the transmit diversity matrix $\boldsymbol{R}_T$ is the covariance matrix between the signals which come from m transmit antennas fed by the same input. Hence, $\boldsymbol{R}_T$ is the covariance matrix associated with the spectral density $\underline{c}(\nu)$ and the set of covariance lags corresponding to the relative distances $\{\tau_k\}_{k=1}^{m-1}$ between the first and the k-th of m aligned transmit antennas:

$$\begin{aligned} \boldsymbol{R}_T &= \int_{-\infty}^{\infty} \underline{c}(\nu)\, \boldsymbol{E}_m(\nu)\, \boldsymbol{E}_m(\nu)^H\, d\nu, \qquad (11) \\ \boldsymbol{E}_m(\nu) &= [1, e^{\imath 2\pi\, \tau_1\, \nu}, \ldots, e^{\imath 2\pi\, \tau_{(m-1)}\, \nu}]^T. \end{aligned}$$

The relationships (10)-(11) may be readily used to evaluate the transmit diversity matrix $\boldsymbol{R}_T$ in the case of a linear transmit antenna array and the reception position(s) situated at the broadside. One can show that a moderate displacement from the broadside (as long as the angle in radians between the broadside and the direction to the receiver may be accurately approximated by its sine) yields a corresponding rotation of the eigenvectors whereas the eigenvalues remain unchanged. Hence (10)-(11) give the access to the eigenvalues of $\boldsymbol{R}_T$ and therefore the capacity of link for a wide range of scenarios with linear transmit antenna arrays and scattering profiles specified by $\rho(u)$.

A. *Free space path loss*

Assume that the scattering points have an equal density in the vicinity of each receive antenna starting from certain minimum distance and that the received energy obeys the standard free space path loss of 20 dB per decade. In the absence of a LOS, the density function $\rho(u)$ may be written as follows:

$$\rho(u) = \frac{u_o}{u^2}\, \mathbf{1}_{u>u_o}, \tag{12}$$

where $u_o = \ell_o/L$ is a normalized version of the minimum distance ℓ_o. The corresponding $\underline{c}(\nu)$ defined in (10) verifies

$$\underline{c}(\nu) = \frac{\sigma_s^2}{\pi}\,\frac{u_o}{\nu^2}\left(1 - \sqrt{1-(\nu/u_o)^2}\,\mathbf{1}_{|\nu|<u_o}\right). \tag{13}$$

The above expression assumes that the receiver is located at the broadside of the transmit antenna array. However, as explained earlier in this section, the eigenspectrum of $\boldsymbol{R}_T$ and therefore the capacity is not so sensitive to moderate variations of the direction to the receiver. The corresponding transmit diversity matrix may be obtained via replacing $\underline{c}(\nu)$ by $\underline{c}(\nu-\nu_o)$ where ν_o corresponds to the angular offset from the broadside of the antenna array.

The case of a present LOS deserves some more attention. In practice, the contribution of scattered signal to the receiver is vanishingly small compared to the contribution of the direct path. This leads to a bad transmit diversity. Indeed, the disparity of propagation paths coming from different transmit antennas is determined by angular separation of the transmit antennas seen by the receive antennas rather than by the area of dominant scattering. When the distance between the receive antennas is small compared to the (average) spatial spread of scattering points, the transmit diversity is worse in presence of a LOS.

A lack for diversity in the presence of a LOS is, however, compensated by a very high SNR when compared to a scenario with no LOS where the link is maintained due to a local scattering only. A significant difference in SNR results in bigger capacities in the presence of a LOS, and this despite bad diversity. Therefore, the system that is designed to operate in a rich scattering environment with no LOS, is expected to perform well enough in the presence of a LOS, due to a high SNR.

V. Numerical example

Let us compare transmit diversity to SDMA in the absence of the LOS. First of all, the accuracy of the stochastic capacity bound C_* and its deterministic approximation C_∞ is studied in transmit diversity and SDMA scenarios. For this purpose, a simplified *physical* model of the signal propagation has been used. Consider a wireless LAN uplink channel in indoor (building) environment. The access point is equipped with 6 receive antennas ($M = 6$) placed 2 m over the ground level. In the transmit diversity scenario, a wireless terminal, equipped with 4 transmit antennas ($m = 4$), is situated 30 m far from the access point ($L = 30$ m), 1 m over the ground level, 10° away from the receive antenna array broadside. The transmit and the receive antennas are linear equispaced arrays with the total length 40 cm ($d = 40$ cm) and 1 m correspondingly. In the SDMA scenario, 4 wireless terminals ($m = 4$) are uniformly spaced around the access point, also 30 m away from the receive antennas. Assume that there is no LOS which often happens in big indoor areas with semi-isolated working spaces. The scattering environment is modeled by a finite number of elementary scattering points that are uniformly distributed in the horizontal plane of the receiver with minor vertical fluctuations. The attenuation of different scattering points varies according to the log-normal law with the mean square attenuation 2 dB. Under the free space loss assumption, the magnitude of each wave is inversely proportional to the propagation distance. The carrier frequency is 5.2 GHz.

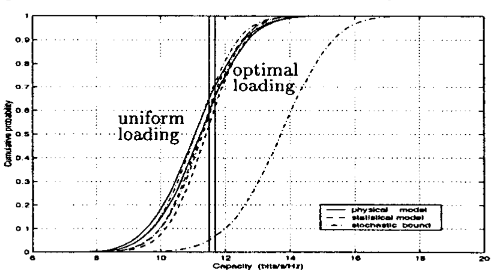

Fig.2. Cumulative probability of the channel capacity: TX diversity, $\varrho^2 = 10$ dB, $M = 6$, $m = 4$.

The cumulative probability distribution of the capacity is studied in the described above transmit diversity and SDMA scenarios, see Fig.2 and Fig.3 correspondingly. The average SNR is fixed to $\varrho^2 = 10$ dB. The solid lines (–) stand for the empirical cumulative probabilities computed from 10000 random trials of the physical model. The dashed lines (– –) stand for the empirical cumulative probabilities corresponding to 10000 random trials of the statistical model specified by (3)-(4). Here $\boldsymbol{R}_R = \boldsymbol{I}_M$ and $\boldsymbol{R}_T$ is computed as explained in the previous section. Finally, the dash-dotted lines (– · –) represent the empirical cumulative probability functions computed from 10000 trials of the stochastic bound

(5) whereas the vertical lines stand for the corresponding C_∞. These results are obtained for the optimal power loading (*i.e.*, with the optimally chosen $\boldsymbol{R}_s$, see [6] for more details) and the uniform loading, with $\boldsymbol{R}_s = (1/m)\boldsymbol{I}_m$. In Fig.2, the dash-dotted line $(-\cdot-)$ stands for the stochastic "*i.i.d.* bound", *i.e.*, the bound from [2] obtained under the assumption of a fully uncorrelated Rayleigh fading. The latter assumption is applied in the SDMA scenario (Fig.3) where it stems from a relatively big random spacing of different terminals. Note that the optimal loading is uniform in the latter case. One can observe a good agreement between the capacity achievable in realistic propagation environment on one hand, and the statistical model as well as its stochastic bound on the other hand. For the transmit diversity scenario, the difference between these three quantities does not exceed 7% and is less than 2% in mean value. The discrepancy between the physical and the statistical modeling appears to be bigger in the SDMA scenario. However, the approximation error always fits within the 10% margin.

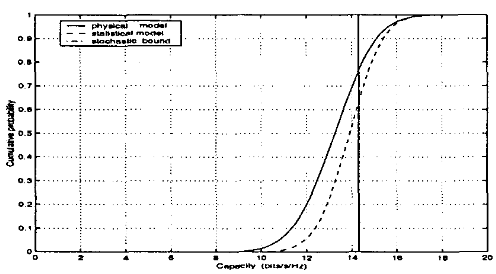

Fig.3. Cumulative probability of the channel capacity: SDMA, $\varrho^2 = 10\,\text{dB}$, $M = 6$, $m = 4$.

Let us analyze the capacity of the multiple antenna channel by means of the stochastic lower bound (5). The following results are obtained from 100000 trials for the described set of parameters unless otherwise is specified. The capacity values correspond to the outage rate 10^{-2}. In Fig.4 through Fig.6, the curves $(-\square-)$ and $(-\nabla-)$ stand for the capacity bounds corresponding to the optimal and the uniform power loading respectively.

According to Fig.4-Fig.6, the transmit diversity schemes yield around 20% loss of a wireless link capacity that may be achieved by SDMA user multiplexing in typical indoor WLAN environments. Also, a relatively small discrepancy between the optimal and uniform power loading methods highlights robustness of the uniform loading. The advantage of a non-uniform power loading becomes important in bad diversity conditions (here big L) and also when most of the link capacity is to be reached by using a big number of transmit antennas.

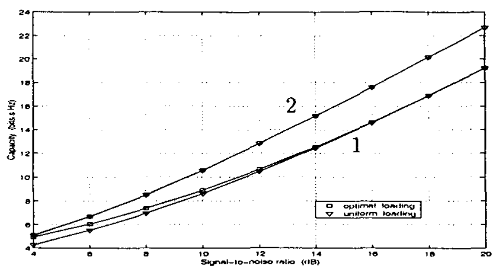

Fig.4. Capacity versus the signal-to-noise ratio ϱ^2: (1) - TX diversity, (2) - SDMA.

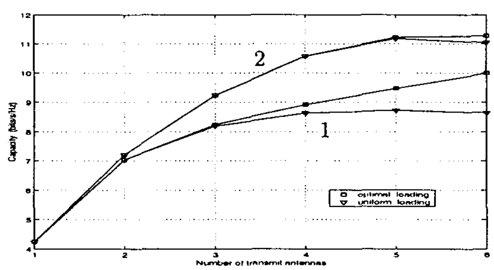

Fig.5. Capacity versus the number of transmit antennas m: (1) - TX diversity, (2) - SDMA.

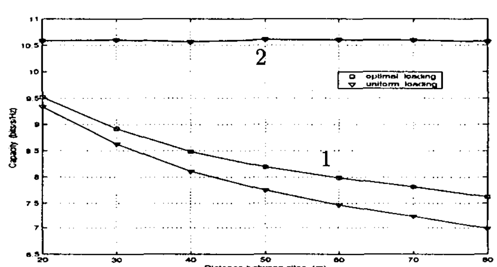

Fig.6. Capacity versus the distance L between TX and RX: (1) - TX diversity, (2) - SDMA.

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